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NATURAL CONVECTION BOUNDARY LAYER FLOW OVER HORIZONTAL AND SLIGHTLY INCLINED SURFACES

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Abstract— Laminar natural convection boundary-layer flow above horizontal and slightly inclined surfaces is considered. The effect of a small surface inclination on flow and transport is studied as a perturbation of flow over a horizontal surface. Solutions of the higher approximations have been numerically determined for the velocity, pressure and temperature fields for a Prandtl number of 0·7. These distributions are also given for the perfectly horizontal surface, for a wide range of Prandtl number. Both the surface conditions of uniform temperature and heat flux are considered. A Mach–Zehnder interferometer was used to study the nature of the laminar boundary-layer. The temperature distributions and heat transfer parameters were measured for the first time in such a flow, in air at atmospheric pressure.

Present results may have a direct application to micro-meteorological and technological problems, and should clarify many aspects of natural convection flows which arise above horizontal and inclined planes through the initial impetus of unstably stratified fluid layers near the surface.

INTRODUCTION

ALTHOUGH natural convection flows over horizontal or nearly horizontal surfaces with prescribed surface conditions have received some attention in the past, they have not been as extensively studied as flows adjacent to vertical surfaces and in freely rising plumes. Micrometeorology is perhaps the field of applied science that has the greatest interest in the subject of the natural convection flows adjacent to nearly horizontal planes bounded by an extensive body of fluid above. Similar flows also arise in many other natural circumstances and in numerous technological applications.

There have been a number of studies of steady laminar natural convection flow over horizontal surfaces, for both isolated surfaces and for surfaces enclosing a body of fluid. Observations of flows arising from isolated surfaces have shown the existence, close to the surface, of a boundary-layer mode of convection, followed, after a region of instability, by a cellular motion.

Schmidt [1] was apparently the first to experimentally investigate the behavior of the flow near the leading edge and above a flat horizontal surface. It was observed that a boundary layer type of flow might be expected near the leading edge.

The first analysis of flow adjacent to a horizontal boundary was carried out by Stewartson [2] for a semi-infinite surface, that is, a surface with a single leading edge. A sign mistake in the analysis led to an erroneous conclusion regarding the conditions for the existence of a boundary-layer flow on such a surface. This was corrected by Gill *et al.* [3], who showed that the only flow that admits a boundary layer solution for simple boundary conditions is the one generated by a heated surface with a single leading edge facing upward or a similar but cooled surface facing downward. No solutions of the governing differential equations were given.

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Kierkus [4] used a perturbation analysis to study the natural convection flow adjacent to inclined isothermal and finite-length surfaces, using the classical boundary-layer solution adjacent to vertical surfaces as the primary flow approximation. Comparison of the calculated results with the experimental data confirmed the analysis.

Rotem and Claassen [5] integrated the boundary layer equations for flow over a semi-infinite horizontal surface of uniform temperature. Results were presented for some specific values of Prandtl number and the limits for zero and infinite Prandtl numbers were also given. The experimental observations, obtained with a semi-focusing Schlieren system, clearly indicated the existence of a boundary layer near the leading edge on the upper side of a heated horizontal surface. The surface was insulated on the bottom face. However, it is not clear whether edge flows were prevented on some sides of the heated surface, i.e. whether the boundary layers formed from all four edges of the plate simultaneously.

This paper reports an analysis and experiments concerning flow over horizontal and slightly inclined surfaces. The effects of a small surface inclination are analysed by perturbing flow over a horizontal surface. Two higher order approximations are found for the velocity, pressure and temperature fields by numerical integration. These calculated temperature distributions are then compared with accurate interferometrically measured temperature distributions in the convection region above a heated surface of uniform temperature. The measured temperature distributions are also used to determine the local heat flux at the wall, and the resulting Nusselt numbers are compared with the predictions of the analysis.

There are several peculiar aspects of flows considered here which must be clarified at the outset. These concern the nature of flow at the leading edge of a surface or, alternatively, the origin of the fluid which flows into the transport region and forms the material of the convecting layer. One extreme circumstance is found, for example, with an horizontal surface losing heat from both its top and bottom sides. The convection layer from the bottom flows around the edges of the plate and turns to begin the flow adjacent to the upper side. More fluid is also induced from the distant region. A very different flow arises adjacent to a heated semi-infinite plate (with a single leading edge), perfectly insulated on the bottom. Here all of the fluid in the convection layer has come in directly from the distant region. Inflow of fluid is caused entirely by the lower pressure levels downstream which are induced by the component of the buoyancy force away from the surface. Flows may also arise with combinations of these and other effects. Thus, many different flow configurations are associated with flat surfaces having leading edge characteristics. Not all past studies have been equally careful in relating experimental configurations to analysis and to possible applications of the results.

This paper concerns two-dimensional flows whose principle direction is normal to the single leading edge of horizontal and slightly inclined surfaces. Attention is restricted to boundary conditions which result in boundary region flows which at the leading edge have zero energy, mass or momentum content. Solutions are given in terms of variables which have these characteristics and apply to such flow. We are thus considering convection resulting above a semi-infinite heated surface set flush in a solid of good insulating material. Of course the experimental surface is of finite extent but measures were taken to ensure the origination of flow from a single leading edge.

ANALYSIS

Consider the flow over a semi-infinite surface in an unbounded and uniform ambient fluid at t_∞. The surface is inclined from the horizontal at a small angle θ. The single leading edge of the surface is taken as the origin of the system of coordinates, x is along this surface and y is normal thereto. The governing momentum, energy and continuity equations are simplified by the Boussinesq and by the boundary-layer approximations to the following form:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}+g\beta\Delta t\sin\theta, \quad (1)$$

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial y}+g\beta\Delta t\cos\theta, \quad (2)$$

$$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}=\frac{k}{\rho C_p}\frac{\partial^2 t}{\partial y^2}, \quad (3)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad (4)$$

where p is called the dynamic or motion pressure, the hydrostatic pressure having been combined with the body force to form the buoyancy force, and $\Delta t = t - t_\infty$. For the system of coordinates chosen, the signs of the two buoyancy force components above apply for the flow adjacent to a heated surface facing upward (or to the equivalent problem of a cooled surface facing downward).

A solution in similarity form is possible for the above complete set of differential equations only when the buoyancy force term in the x-momentum equation (1) is zero, i.e. for $\theta = 0$. Then the velocity, temperature and motion pressure fields are expressible in a single variable $\eta(x, y)$. This is the problem of Stewartson [2] and Gill *et al.* [3]. Numerical solutions were given by Rotem and Claassen [5] for a uniform temperature surface condition for Prandtl numbers ranging from 0·1 to 10.

Since for a horizontal surface there is no component of buoyancy force parallel to the surface, the flow is driven entirely by a gradient in the dynamic pressure. This arises from the normal component of buoyancy and is negative. However, for surfaces even slightly inclined to the horizontal, the parallel component of buoyant force may not be neglected. This effect may be retained by a perturbation analysis and the resulting perturbation parameter is a function of surface inclination.

The analysis below reduces the equations above to sets of three ordinary differential equations, one set for each order of approximation considered. A similarity variable $\eta(x, y)$ is introduced. The stream function $\psi(x, y)$ is expanded in terms of the perturbation parameter $\varepsilon(x)$ and stream functions $f_1, f_2, \ldots$.

$$\eta = \frac{y}{x}\sqrt[5]{\left(\frac{Gr_x}{5}\right)}, \quad (5)$$

$$\psi = 5\nu\sqrt[5]{\left(\frac{Gr_x}{5}\right)}\{f_0(\eta)+\varepsilon(x)f_1(\eta)+\ldots\}, \quad (6)$$

where the local Grashof number is defined as

$$Gr_x = \frac{g\beta(t_0 - t_\infty)\cos\theta\, x^3}{\nu^2}. \quad (7)$$

The surface temperature (t_0) distribution is initially assumed to depend on x as Nx^n, N being a positive constant for any given circumstance so that

$$t_0 - t_\infty = Nx^n. \quad (8)$$

The dimensionless temperature function $\Phi(\eta, x)$ is defined as

$$\Phi(\eta, x) = \frac{t - t_\infty}{t_0 - t_\infty} = \Phi_0(\eta) + \varepsilon(x)\,\Phi_1(\eta) + \ldots \quad (9)$$

Therefore

$$t - t_\infty = Nx^n\Phi(\eta, x). \quad (10)$$

The appropriate expression for the dynamic pressure variation $p(x, \eta)$ is imposed by the y-momentum equation (2) as

$$p = \frac{5\nu^2\rho}{x^2}\sqrt[5]{\left(\frac{Gr_x}{5}\right)^4} \times \{G_0(\eta)+\varepsilon(x)\,G_1(\eta)\ldots\}, \quad (11)$$

where the G are pressure functions in terms of η.

The condition that the x-component of the body force term should appear initially in the first order correction is used to determine the value of the perturbation parameter $\varepsilon(x)$, as

$$\varepsilon(x) = \sqrt[5]{\left(\frac{Gr_x}{5}\right)}\tan\theta. \quad (12)$$

The nature of the above expansion in terms of the above parameter $\varepsilon(x)$ needs some justification in relation to the usual approximations of boundary layer theory. The flow considered here may be described in terms of an expansion of the complete equations, if leading edge effects are omitted, in terms of the two parameters

$$\xi = \frac{1}{\sqrt[5]{(Gr_x)}} \quad \text{and} \quad \varepsilon = \sqrt[5]{\left(\frac{Gr_x}{5}\right)} \tan\theta,$$

Therefore, a distribution function, say f, would be written as follows:

$$f(\eta) = f_0(\eta) + \xi F_1(\eta) + \varepsilon f_1(\eta) + \dots$$

Ordinary boundary-layer theory applies for flows sufficiently vigorous that $\xi \ll 1$, so that the corrections in ξ may be neglected. However, this approximation is justified in considering the effects of surface inclination alone when the second parameter ε is appreciably larger than ξ. The ratio of the two is

$$\frac{\varepsilon}{\xi} = \left[\frac{Gr_x}{5^{\frac{1}{2}}}\right]^{\frac{2}{5}} \tan\theta.$$

For a given surface inclination, the region (in x) where the previous relation applies can be determined. Our experimental studies are in the range 3–13 for this ratio. The lower value is a consequence of the need to stay away from the surface leading edge, so that boundary-layer approximations are admissible. The upper value of the ratio appears from our studies to result inevitably for flow conditions at small θ in air that are sufficiently stable to be considered attached laminar flow.

The continuity equation (4) is satisfied by ψ and the remaining equations (1), (2) and (3) are transformed by (5), (6), (10) and (11). The resulting system of equations for the flow to be perturbed, applicable to a heated surface facing upward, are

$$f_0''' + (3 + n)f_0''f_0 - (2n + 1)f_0'^2 - \frac{n-2}{5} G_0'\eta - \frac{4n+2}{5} G_0 = 0, \quad (13)$$

$$G_0' = \Phi_0, \quad (14)$$

$$\Phi_0'' + Pr\,\{(n + 3)\,\Phi_0' f_0 - 5n\Phi_0 f_0'\} = 0. \quad (15)$$

The equations for next order of approximation (f_1, ϕ_1, G_1), in terms of $\varepsilon(x)$, are

$$f_1''' + (3 + n)f_1''f_0 - 5(n + 1)f_0'f_1' + 2(n + 3)f_1 f_0'' - (n + 1)\,G_1 - \frac{n-2}{5} G_1'\eta + \Phi_0 = 0, \quad (16)$$

$$G_1' = \Phi_1, \quad (17)$$

$$\Phi_1'' + Pr\,[(3 + n)f_0\Phi_1' - (3 + 6n)f_0'\Phi_1 - 5nf_1'\Phi_0 + 2(3 + n)f_1\Phi_0'] = 0. \quad (18)$$

The equations for the next higher approximation (f_2, ϕ_2, G_2), in terms of $\varepsilon^2(x)$, are given below. These are not important for their contribution to the solution, since the magnitude of this correction may become, for very small surface inclinations, of the same order of magnitude as the boundary layer approximations already made.

$$f_2''' + (3 + n)f_2''f_0 - 2(3n + 4)f_2'f_0' + 3(n + 3)f_2 f_0'' + 2(n + 3)f_1''f_1 - (3n + 4)f_1'^2 - \frac{6n+8}{5} G_2 - \frac{n-2}{5} G_2'\eta + \Phi_1 = 0, \quad (19)$$

$$G_2' = \Phi_2, \quad (20)$$

$$\Phi_2'' + Pr\,[(n + 3)f_0\Phi_2' - (7n + 6)f_0'\Phi_2 - (6n + 3)f_1'\Phi_1 + 2(n + 3)f_1\Phi_1' + 3(n + 3)\,\Phi_0' f_2 - 5n\Phi_0 f_2'] = 0. \quad (21)$$

Appropriate values of n, the temperature condition, along with boundary conditions, are still to be specified. The applicable boundary conditions for a prescribed surface temperature condition Nx^n arise from physical restrictions at the wall and in the distant medium. For the unperturbed flow these are:

$$f_0(0) = f_0'(0) = f_0'(\infty) = \Phi_0(\infty) = G_0(\infty) = 0, \text{ and } \Phi_0(0) = 1. \quad (22)$$

Boundary conditions for the next two higher orders are respectively

$$f_1(0) = f'_1(0) = \Phi'_1(0) = f'_1(\infty) = \Phi_1(\infty) = G_1(\infty) = 0, \quad (23)$$

and

$$f_2(0) = f'_2(0) = \Phi'_2(0) = f'_2(0) = \Phi_2(\infty) = G_2(\infty) = 0. \quad (24)$$

The total rate of thermal energy convected in the boundary layer at any location x is calculated as

$$Q = 5\rho C_p \nu N I \left[\frac{g\beta N \cos\theta}{5\nu^2}\right]^{\frac{1}{4}} x^{\frac{6n+3}{5}} \quad (25)$$

where I is the value of the following integral:

$$I = \int_0^\infty \Phi f' \mathrm{d}\eta = \int_0^\infty \{\Phi_0(\eta) + \varepsilon(x)\Phi_1(\eta)\ldots\}. \{f'_0(\eta) + \varepsilon(x) f'_1(\eta)\ldots\}\,\mathrm{d}\eta. \quad (26)$$

For horizontal surfaces $[\varepsilon(x) = 0]$, the integral I is the solely result of the similarity solution f_0 and Φ_0 and is, therefore, dependent only upon the Prandtl number and n. Thus, the value of the exponent n may be found from (25) after imposing any desired condition on the way in which Q varies with x. For an uniform flux surface, Q must increase linearly with x and $n = \frac{1}{3}$. The condition that Q be independent of x requires $n = -\frac{1}{2}$ (this is the condition when all the convected energy is being released by a line source at the leading edge of the surface). For other surface conditions, corresponding values of n may be determined. It may be seen from (8) that $n = 0$ specifies a surface of uniform temperature.

For inclined surfaces $[\varepsilon(x) \neq 0]$ the integral I in equation (26) is apparently a function of the solutions from the various orders and is thus x dependent. Therefore, a uniform surface heat flux does not result for $n = \frac{1}{3}$ when higher levels of approximation are considered. Uniform or other prescribed surface temperature conditions are not affected since no conditions on the heat flux are implied. The coefficient N is related to the total convected heat by (25) for any desired surface condition.

Equations (13)–(21) were numerically integrated, subject to boundary conditions (22)–(24), across the boundary layer for a uniform surface temperature ($n = 0$). Solutions for the uniform flux case ($n = \frac{1}{3}$) were also obtained for a horizontal surface $[\varepsilon(x) = 0]$ by integrating (13)–(15), subject to the boundary conditions (22). A range of Prandtl number was considered for each condition. Various representative temperature and velocity distributions are given in Figs. 3–9. These results are discussed later.

The experiments of Rotem and Claassen [5] indicated that the flow over a flat plate remains attached to the surface only a certain distance from the leading edge, after which it appeared to separate. We decided to experimentally determine the extent of the validity of the above numerical results and to study the question of flow attachment. These experiments and the principal results therefrom are described below.

THE EXPERIMENT

The experiment was designed to determine the detailed nature of a boundary region flow over horizontal and slightly inclined heated surfaces and to compare the experimentally determined temperature distributions and surface heat transfer with those computed above.

The experiment was carried out in air at atmospheric pressure. Figure 1a shows the design of a plate assembly used. It consists of a 14 in. wide by 17 in. long aluminum plate of 1 in. thickness. It was electrically heated from below with twelve independent strip heaters, parallel to the leading edge. The heaters were 1 in. wide and equally spaced. They were bonded to the plate with high temperature cement. Each heated segment was provided with a thermocouple probe within 0·010 in. of the top surface. A uniform temperature on the surface of the plate was achieved by adjusting the electrical input to each of the independent heaters.

The analysis of the preceding section indicated

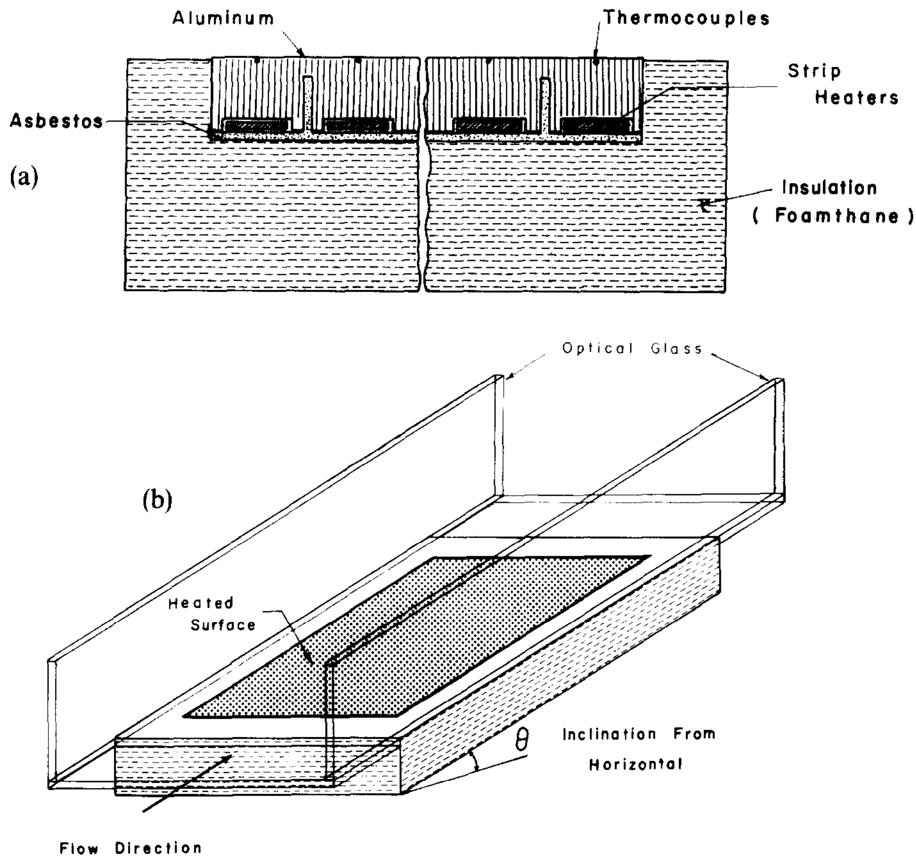
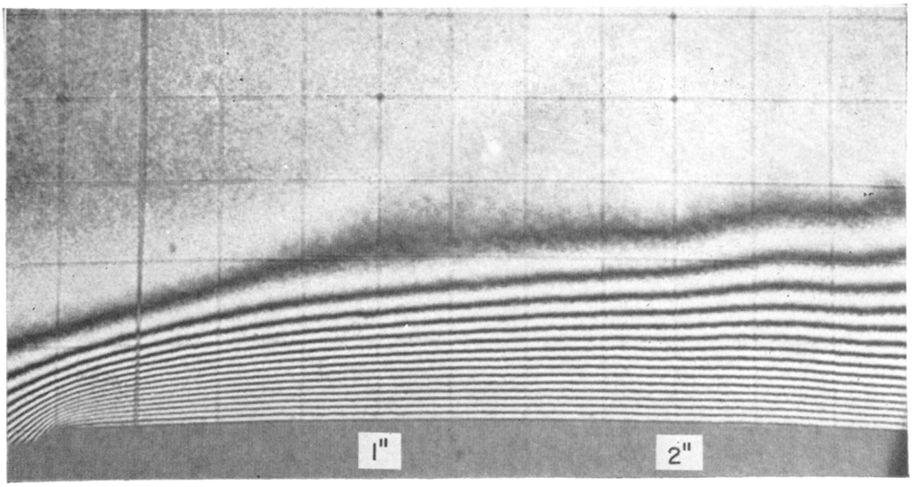


FIG. 1. Diagram of the uniform temperature plate set-up. (a) Section showing heaters, groove, and insulation. (b) General view.

that, for $\theta = 0$, the heat released by the plate must increase as the $\frac{3}{5}$ power of the distance from the leading edge to achieve a uniform surface temperature condition. Good thermal conduction along the plate, in the x direction, would make the establishment of such a variation of the heat flux difficult. Therefore, deep grooves were opened from the bottom of the plate between adjacent strip heaters. This arrangement greatly reduced the heat conduction in the flow direction, while maintaining the uniformity and rigidity of the surface of the plate.

The plate assembly was insulated on the bottom and four sides to ensure that the heat input was mainly dissipated at the upper free surface of the plate, thus avoiding a flow which would otherwise arise on the four sides and over the bottom of the plate. The twelve copper–constantan thermocouple probes near the surface allowed adjustment for surface temperature uniformity. They also served as a verification of the evaluation of interferograms in terms of index of refraction. The agreement between measured temperatures and those inferred from interferograms was always better than 0·5°F. All thermocouples were initially calibrated in a constant temperature oil bath with precision mercury-in-glass thermometers. The thermocouple junctions were embedded in epoxy resin in small holes drilled upward from the lower

(2a)



(2b)

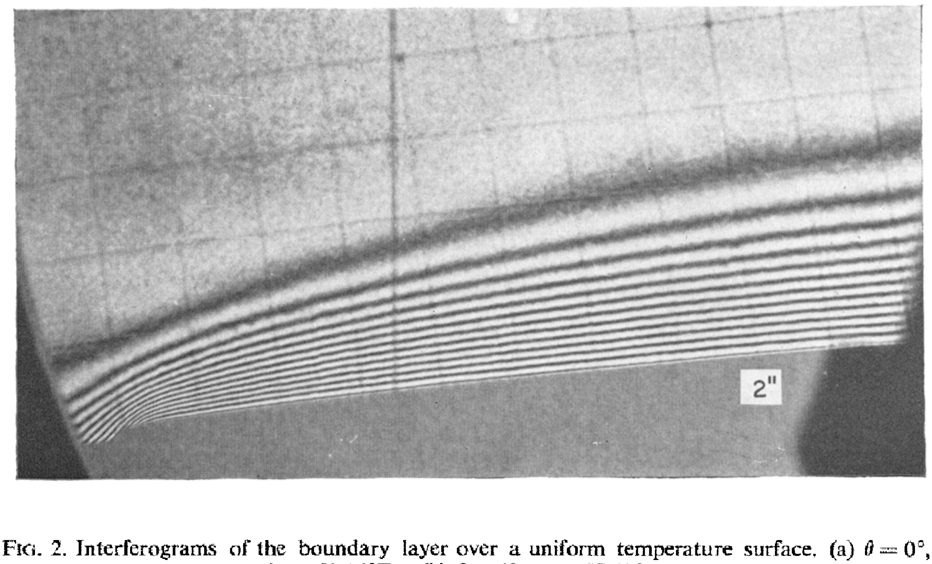


FIG. 2. Interferograms of the boundary layer over a uniform temperature surface. (a) $\theta = 0°$, $\Delta t = 53{\cdot}16°F$, (b) $\theta = 6°$, $\Delta t = 57{\cdot}49°F$.

side of the plate. The upper surface of the plate was accurately machined and polished to eliminate flow disruption due to surface imperfections.

The plate could be easily maintained uniform in temperature t_0 at a level. For example, for $t_0 - t_\infty = 65°F$ the uniformity was within 0·1°F. The output of the thermocouples were measured with a precision potentiometer (LN K-3). The insulated and vibration mounted plate could be inclined to any prescribed small angle within 0·2 degree accuracy.

To generate essentially a two-dimensional flow, to prevent incoming flow from the sides of the plate, optical quality glass windows were located on both sides of the heated surface, as shown in Fig. 1b. The flow from the leading and trailing edges was not impeded in any ascertainable way.

A 5 in. aperture Mach–Zehnder interferometer was used to determine the temperature field under various conditions of surface heating and inclination. The light source was a Mercury vapor lamp with a green interference filter. The interferometer sensitivity was approximately 3·6 degrees per fringe for the two-dimensional field 14 in. wide. Adjustment was made to the infinite fringe so that each fringe of the interferogram represented approximately an isothermal contour. The interferograms were recorded on photographic film and were interpreted with an optical comparitor which permitted a 50 × magnification.

Figure 2 reproduces two typical interferograms of the temperature field. The rectangular grid seen was introduced to assess optical distortions introduced in the field by the auxiliary optical system and for distance estimation. A second grid, accurately calibrated but not shown in the photographs, was positioned in the center of the plate (in the focal plane). This grid served as a frame of reference for more accurate measurements of fringe location. After an exposure of this reference grid was taken at the beginning of the experiment it was removed from the field. The spacing between lines in both of these grids was $\frac{1}{4}$ in. Interferograms were taken with the plate held horizontal and with inclinations of 2, 4 and 6 degrees from the horizontal and were restricted to the region of about 4 in. from the leading edge. Preliminary investigation had indicated that this was the region in which simple attached boundary layer flow persisted.

The relation between temperature difference and fringe shift is given by the Dale–Gladstone equation

$$\frac{\Delta t}{N} = \frac{R\lambda t_\infty^2}{Wkp}\left\{1 + \frac{\Delta t}{t_\infty}\right\} \tag{27}$$

where k is the Dale–Gladstone constant, W is the width of the field (assumed to be equal to the width of the plate) p is the local atmospheric pressure, R is the universal gas constant, t_∞ is the far field temperature as measured by a mercury-in-glass thermometer during the experiment and λ is the wavelength of the monocromatic (5460 Å) light source of the interferometer. For gases at low pressure, as for the present case, the above expression cannot be reduced to a simpler explicit relation between fringe shift and temperature difference without accepting large errors in the evaluation of the temperature field, larger than 10 per cent. The fringe-temperature factor is therefore not a simple constant. In this experiment it was evaluated at each fringe location by an iterative procedure.

RESULTS AND CONCLUSIONS

The three coupled equations which govern flow over a semi-infinite horizontal surface, (13)–(15), were numerically integrated, with boundary conditions (22), using a fourth-order Runge–Kutta method across the boundary region. Contrary to the experience of Rotem and Claassen [5], the process of numerical integration was found to be stable and rapidly convergent. It did not require any special numerical techniques for either the uniform temperature or uniform flux conditions analyzed.

The results of the integrations, i.e. the velocity, pressure and temperature functions for the uniform flux horizontal surface are shown on Figs. 3–5 over the Prandtl number range from 0·1 to 10. These curves show the decrease of velocity and thinning of the thermal layer, with increasing Prandtl number, that we are accustomed to in vertical flows. However, the velocity effect appears to arise in a different way

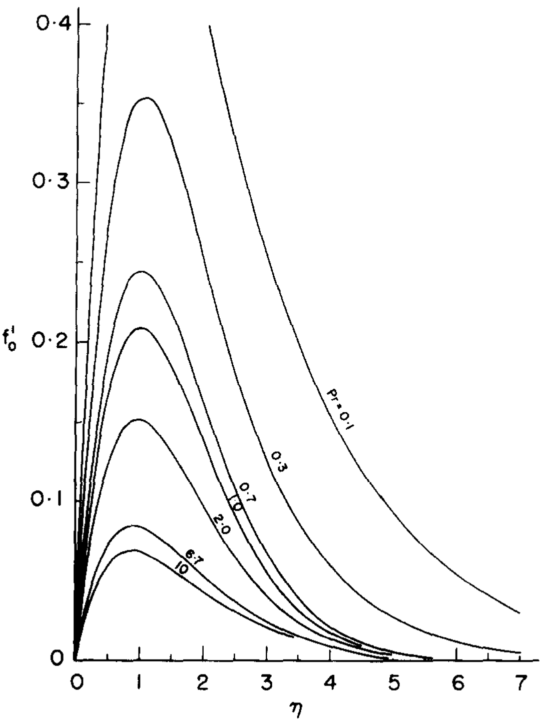


FIG. 3. Computed velocity function for a range of Prandtl number. Uniform flux horizontal surface.

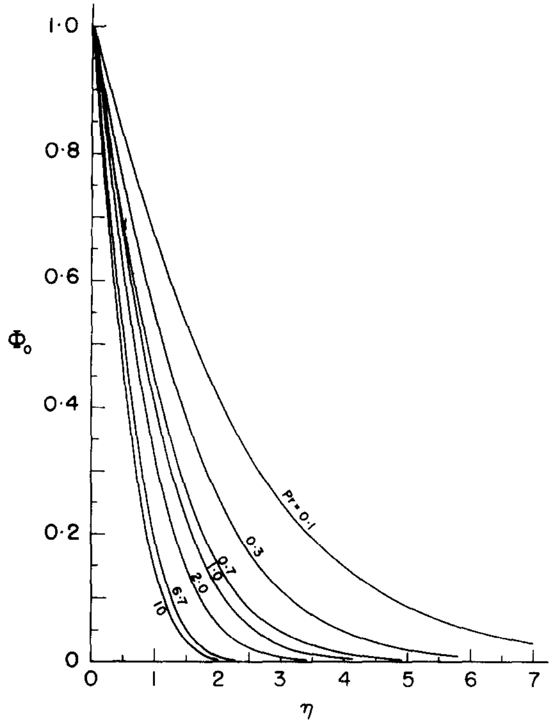


FIG. 4. Computed temperature function for a range of Prandtl number. Uniform flux horizontal surface.

Table 1. Results of calculations

Pr	$f(\eta)$	$f''(0)$	$G(0)$	$\Phi'(0)$
		Uniform flux horizontal surface		
0·1	1·66641 (η = 12)	1·16319	−2·1265	−0·36075
0·3	0·89999 (η = 12)	0·767089	−1·4464	−0·50250
0·7	0·54740 (η = 7)	0·54990	−1·10162	−0·63091
1·0	0·46634 (η = 7)	0·47875	−0·99008	−0·69184
2·0	0·33804 (η = 7)	0·36528	−0·81516	−0·81963
6·7	0·18825 (η = 5)	0·22895	−0·60234	−1·08201
10·0	0·15750 (η = 5)	0·19615	−0·54842	−1·18237
		Uniform temperature horizontal surface		
0·7	0·64613 (η = 7)	0·51910	−1·26831	−0·48905
0·7*	0·37054 (η = 7)	0·37674	0·23560	−0·10426
0·7†	0·00102 (η = 7)	−0·01032	−0·06321	0·013609

* First order correction for surface inclination.
† Second order correction for surface inclination.

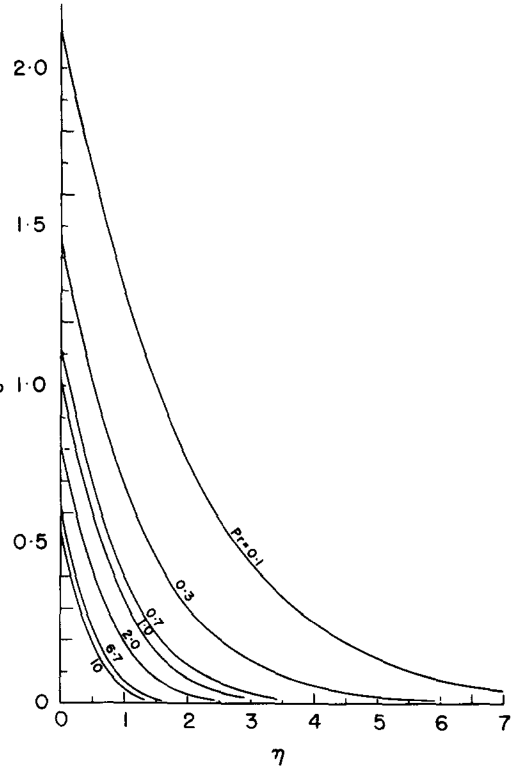


FIG. 5. Computed pressure function for a range of Prandtl number. Uniform flux horizontal surface.

in this flow. The thinning of the thermal layer, seen in Fig. 4, results in the large decrease in the motion-driving pressure gradient seen in Fig. 5.

Calculated values of the above functions at the boundaries are listed in Table 1. These were obtained with an iterative procedure which permitted numerical corrections to initial arbitrary guesses of such values. Results for the uniform temperature horizontal surface may be found in Rotem and Claassen [5].

For slightly inclined surfaces, the first and second order approximation equations, (16)–(21), were also integrated with corresponding boundary conditions (23) and (24), for a uniform temperature surface ($n = 0$) and for $Pr = 0{\cdot}7$. Calculated values are also listed in the table. The behavior of the velocity and temperature functions are compared in Figs. 6 and 7. Composite solutions of velocity and temperature are seen in Figs. 8 and 9 for values of the perturbation parameter $\varepsilon(x)$ of 0, 0·5 and 1·0. These values correspond, for example, to angles of inclination of zero and about 3 and 6 degrees for $Gr_x = 5 \times 10^5$. Very small inclinations of the surface are seen to have a large affect on the velocity distribution but a small one on the temperature distribution. Thus, inclination would be expected to strongly affect laminar stability. We note that the second order approximation is very small.

The surface heat transfer parameter $Nu/\sqrt[5]{(Gr_x)}$ is plotted vs Prandtl number in Fig. 10 for a horizontal surface. Values are given for both the uniform temperature and uniform heat flux surface conditions over the range of Prandtl number considered. These results are in agreement with those calculated by Rotem and Claassen [5] for the uniform temperature surface and Prandtl numbers of 0·1–10. Our result for $Pr = 0{\cdot}7$ and uniform flux also agree

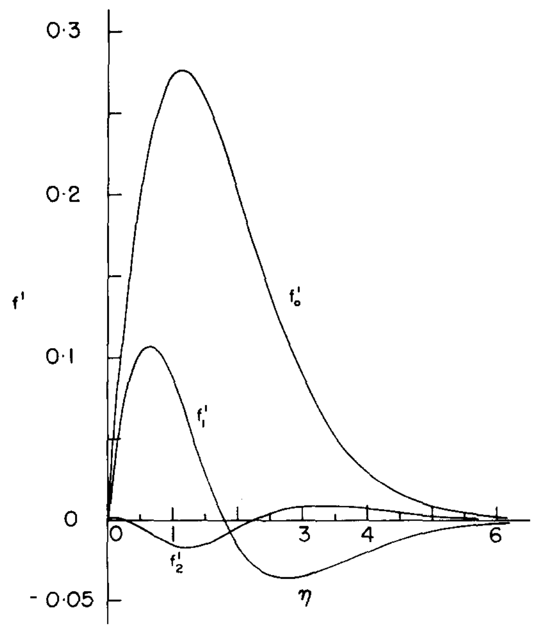


FIG. 6. f_0, f'_1 and f'_2 velocity functions. Uniform temperature surface. $Pr = 0{\cdot}7$.

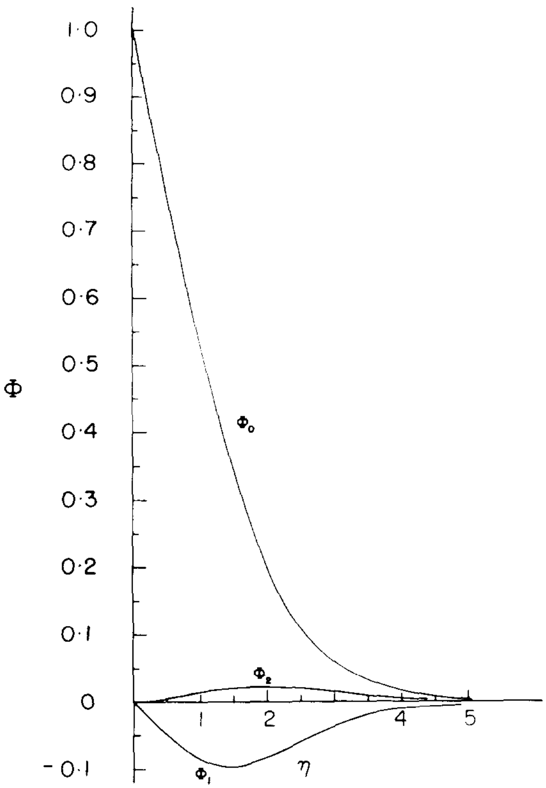


FIG. 7. Φ_0, Φ_1 and Φ_2 temperature functions. Uniform temperature surface. $Pr = 0{\cdot}7$.

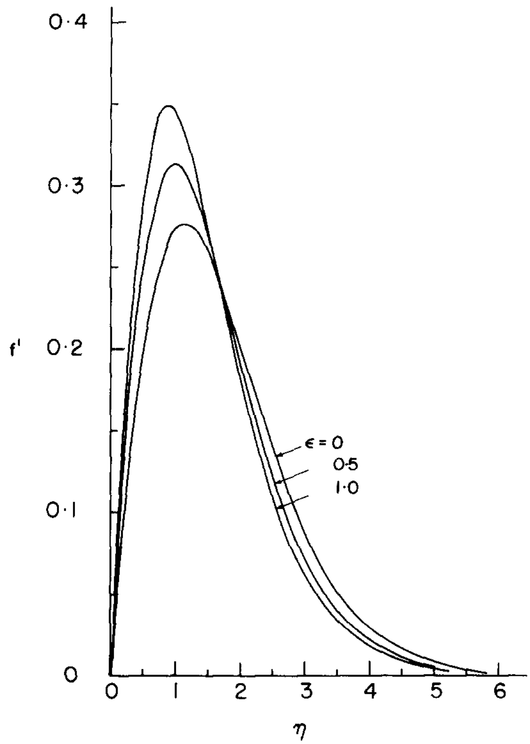


FIG. 8. Effect of inclination on the velocity function. Uniform temperature surface. $Pr = 0{\cdot}7$. For example: $\varepsilon = 0{\cdot}5$ and $1{\cdot}0$ for inclinations of 3° and 6° respectively for $Gr_x = 5 \times 10^5$.

with the calculation by H. K. Kuiken reported by Rotem and Claassen [5].

The results in Fig. 10 show that the local Nusselt number is approximately proportional to the fourth root of the Prandtl number, in the range of values considered, for both the uniform temperature and uniform flux surface conditions. For Prandtl numbers around 1·0 the local Nusselt number is given very closely by

$$Nu = \frac{hx}{k} = 0{\cdot}394 \sqrt[5]{(Gr_x)} \sqrt[4]{(Pr)}$$

Uniform temperature surface (28)

$$Nu = \frac{hx}{k} = 0{\cdot}5013 \sqrt[5]{(Gr_x)} \sqrt[4]{(Pr)}$$

Uniform flux surface. (29)

Note that the Grashof number appears as the fifth root. Thus the heat transfer to the boundary layer region adjacent to a horizontal semi-infinite plate is less than from a vertical plate for the same fluid and temperature conditions. For example, the heat transfer at the location where the local Grashof number of the flow is 10^5 is about 80 per cent higher for a vertical surface than for a horizontal one.

These correlations over-estimate the heat transfer for large and small values of the Prandtl number. At $Pr = 100$ the over-estimation is about 12 per cent and it amounts to 16 per cent at $Pr = 0{\cdot}1$. The $\sqrt[4]{(Pr)}$ approximation is similar to that used for vertical natural convection adjacent to isothermal surfaces.

The effect of surface inclination, $\varepsilon(x)$, on local heat transfer is seen as a solid line in Fig. 11 for $Pr = 0{\cdot}7$. Recall that inclination increases

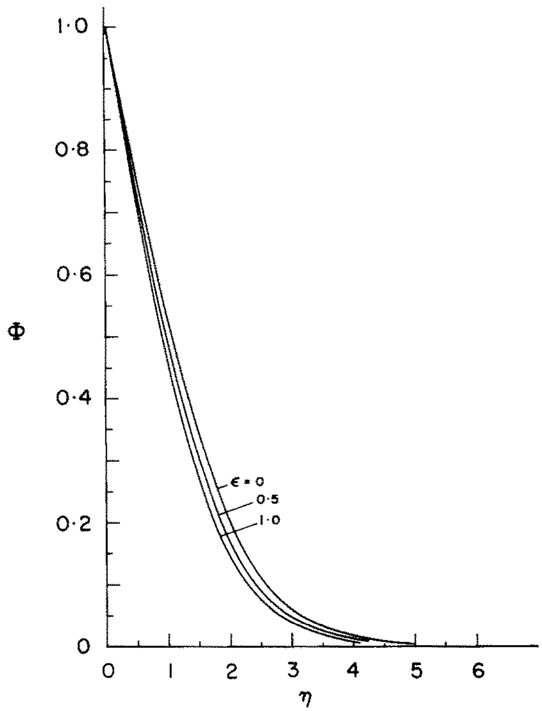


FIG. 9. Effect of inclination on the temperature function. Uniform temperature surface. $Pr = 0{\cdot}7$. For example: $\varepsilon = 0{\cdot}5$ and $1{\cdot}0$ for inclinations of 3° and 6° respectively for $Gr_x = 5 \times 10^5$.

with ε at constant Gr_x. Inclination clearly increases the local heat transfer and the calculated increase amounts to about 18 per cent for $\varepsilon = 1$.

Two examples of the many interferograms recorded for different surface inclinations were previously shown in Fig. 2. The upper picture shows the flow region over an horizontal surface and the lower one the flow adjacent to a surface inclined 6 degrees from the horizontal. It is seen that surface inclination results in a thinner and more regular boundary region flow.

Temperature distributions determined from such interferograms are compared with the computed results in Figs. 12–14 for values of ε of 0, 0·5 and 1·0 respectively. The data was obtained for the range of local Grashof number and plate inclinations specified on the figures. The agreement between the calculated and measured temperature distributions improves drastically with ε but the agreement is good only in a relatively small range of Grashof numbers for $\varepsilon = 0$ and 0·5. All data for $Gr_x < 10^4$ lies above the curves. This probably results from the inaccuracy of boundary layer approximations at these low values. The experimental data for $Gr_x > 10^4$ are somewhat below the predicted

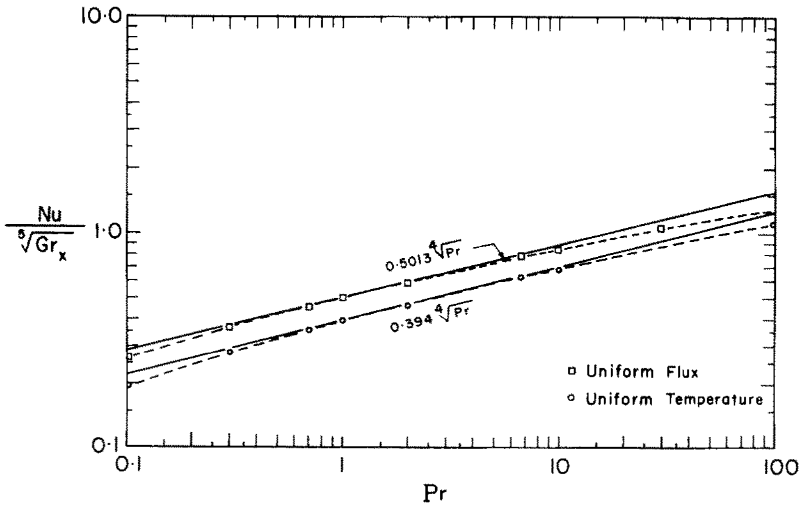


FIG. 10. Local Nusselt number for uniform temperature and uniform flux horizontal surface.

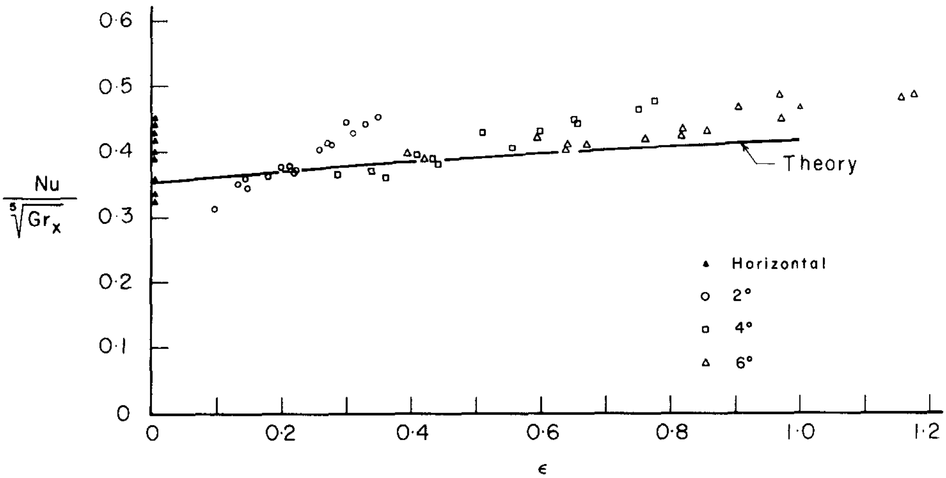


FIG. 11. Effect of the inclination on the local Nusselt number. Uniform temperature surface. Theoretical results, $Pr = 0{\cdot}7$. Experimental data in air at atmospheric conditions. Range of surface inclination 0–6 degrees.

curves, with, therefore, a larger temperature gradient and heat transfer at the surface than predicted.

The trends in this data agree with those calculated. Yet the levels are different and the data is consistently below the calculated curves. One cause of this disagreement is thought to be due to an ambiguity about the location of the leading edge. The upstream edge of the metal surface was taken as $x = 0$. However heat conduction in the insulating material results in an upstream temperature disturbance, as seen in Fig. 2. Had the test fluid been a liquid, with a relatively higher thermal conductivity compared to the insulation, this effect would have been much reduced. However, there is another possible mechanism which may cause the measured temperatures to be systematically lower than the theoretical predictions and it is a very important fundamental aspect of such flows. At some downstream location the boundary layer material becomes the origin of a rising thermal plume, by mechanisms not now well understood. This produces a large buoyancy force, which presumably results in very low dynamic pressure p at its base. Recall than $p \propto x^{-\frac{2}{5}}$ in the boundary layer. This gradient is the driving mechanism of the flow. In the region where vertical flow arises the dynamic pressure level may drop below that estimated for continuously attached and relatively thin boundary-layer flow. This would modify the pressure distribution in the upstream attached flow and amount to an additional driving mechanism. The net result would be a thinner boundary layer, as observed here for all inclinations. For the horizontal orientation and for small inclination ($\theta < 4$ degrees) the thinning effect is more pronounced due to a relatively shorter (in x) boundary layer region. At higher inclinations the boundary layer flow persists further and the pressure distribution is presumably nearer that of the simularity solution.

The above reasoning also helps explain why the disagreement between experimentally determined and computed values of the local Nusselt number depends on surface inclination. The comparison is seen in Fig. 11. Measured heat transfer rates for both the horizontal and slightly inclined orientations were found to be slightly lower than the calculated ones for $Gr_x < 10^4$ and higher for $Gr_x > 10^4$. The increased gradient of p thins the thermal layer and increases heat

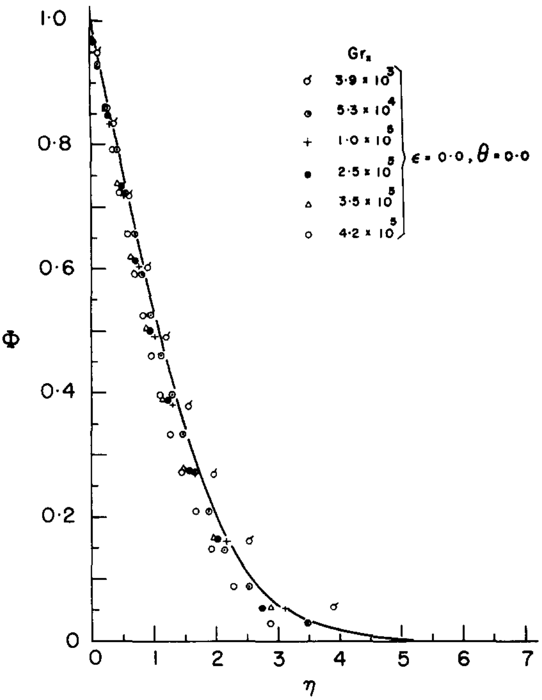


FIG. 12. Temperature distribution over a uniform temperature horizontal surface. Similarity solution for $\varepsilon = 0$. Experimental data in air at atmospheric conditions.

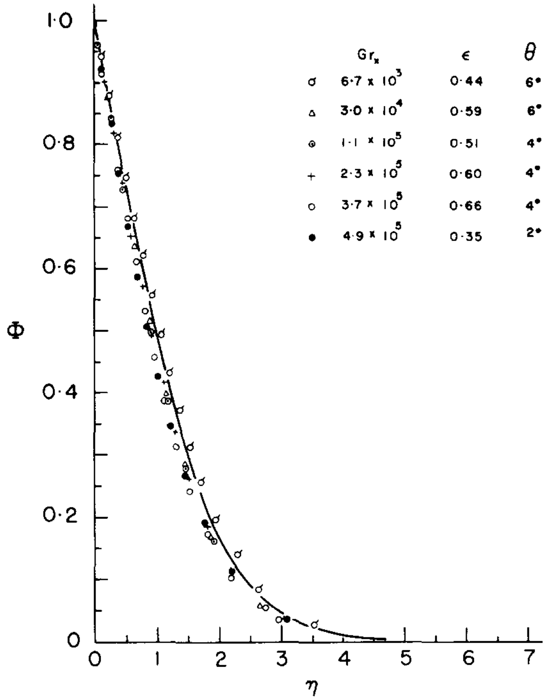


FIG. 13. Temperature distribution over a uniform temperature slightly inclined surface. Similarity solution for $\varepsilon = 0{\cdot}5$. Experimental data in air at atmospheric conditions.

transfer. The trend of the data in Figs. 12–14 is consistent with that of the computed results of Fig. 9.

Since the effect of surface inclination is reflected more strongly in the velocity rather than in the temperature distribution, it would have been a more critical test of the theory to measure velocity distributions in the boundary region. However, most conventional measuring methods for such low velocity levels are quite inaccurate. The hot-wire could, in principle, be calibrated to this level but there are unresolved problems in their calibration for this range.

The experimental results above seem very reliable. In the range of Grashof numbers and inclinations studied the temperature distribution was repeatable to about 1 per cent. Heat transfer results were repeatable to about 5 per cent. The higher inaccuracy in the determination of the heat transfer results arise from the additional approximations in the determination of the slope of the temperature profile at the surface.

The local surface heat transfer rate, calculated from the slope of the temperature profile at the wall as determined by interferometry, were not compared with the electrical input to the heaters. The heat loss to the insulation material around the plate was difficult to compute accurately and the radiation loss from the surface was comparable to the heat convection rate. All fluid properties used in evaluating the Grashof and Prandtl numbers were evaluated at the "film" temperature, i.e. at the average temperature between the surface and the far field.

The agreement between the surface temperature as determined by analysing interferograms and by thermocouple readings was always

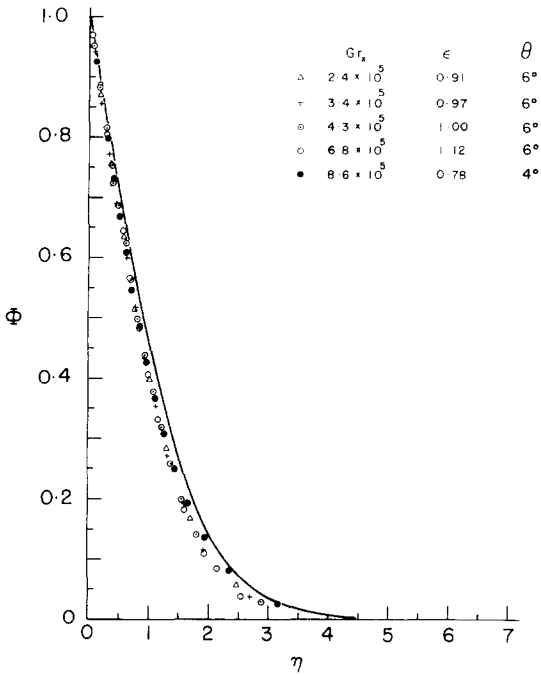


FIG. 14. Temperature distribution over a uniform temperature slightly inclined surface. Similarity solution for $\varepsilon = 1{\cdot}0$. Experimental data in air at atmospheric conditions.

within 0·8°F. This also validates the assumption that the width of the disturbed optical path was equal to the width of the heated plate (14 in.).

ADDITIONAL OBSERVATIONS

The present study does not consider any of the complicated questions of stability which inevitably arise in connection with such flows, nor does it treat the possibility of three-dimensional effects or of possible boundary-layer separation and subsequent cellular convection or plume formation. The color Schlieren photographs presented by Rotem and Claassen [5], as well as the present interferometric study, with additional flow visualization of streamlines by smoke, clearly reveal that severe instability mechanisms disrupt the boundary layer flow some distance from the leading edge. Three-dimensional flow effects arise in the boundary-layer region. Since, for the horizontal and slightly inclined surface, the local Grashof number at which instability was first observed is lower than that observed by Tritton [6] on surfaces at much higher inclinations, one would conclude that increasing inclination of the surface from horizontal enhances stability.

Instability and separation mechanisms become very important in many natural convection applications. An interesting example occurs in a local atmosphere–surface interaction where, for example, solar heating produces thermal instability and surface flows. The separation and/or transition of such flows are related to local up-draft formation. Such circulations are the agency of the diffusion of local gradients and of their general dispersion.

Our flow visualization of streamlines by smoke injection indicated that the onset of the formation of vertical unattached flow is a function of surface inclination. For the horizontal surface, with a temperature difference $t_0 - t_\infty$ of about 60°F, this separation was first observed at a $Gr_x^{\frac{1}{5}}$ of approximately 80. This is in close agreement with the results of Rotem and Claassen [5]. However, for a surface inclination of 12 degrees, the first separation was delayed to $Gr_x^{\frac{1}{5}}$ of approximately 140. We have also found that the value of Gr_x at separation may be reduced by introducing disturbances into the upstream attached flow.

The relation between the kind of flow separation we've seen and the local thermal instability is not clear. We calculated a Rayleigh number, based on the thickness of the boundary layer. The first observed flow separation occured at a Rayleigh number of about 2×10^4 for the horizontal surface and at about $4{\cdot}5 \times 10^4$ for a surface inclined 12 degrees from the horizontal. Mechanisms of instability and of separation are the topic of the following paper.

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ECOULEMENT AVEC COUCHE LIMITE PAR CONVECTION NATURELLE SUR DES SURFACES HORIZONTALES ET LEGEREMENT INCLINEES

Résumé—On considère l'écoulement par convection naturelle avec couche limite laminaire sur des surfaces horizontales et légèrement inclinées. L'effet d'une petite inclinaison de la surface sur l'écoulement et le transport a été étudié comme une perturbation de l'écoulement sur une surface horizontale. On a numériquement déterminé, pour les champs de vitesse, de pression et de température, les solutions de plus grand ordre d'approximation à un nombre de Prandtl de 0,7. De même, ces distributions ont été données pour la surface parfaitement horizontale pour un large domaine du nombre de Prandtl. Les conditions de température et de flux thermique uniformes à la surface ont été considérées.

Afin d'étudier la nature de la couche limite laminaire, on a utilisé un interféromètre de Mach–Zehnder. Les distributions de température et les paramètres de transfert thermique ont été mesurés pour la première fois dans un tel écoulement, dans l'air à la pression atmosphérique.

Les résultats présentés peuvent avoir une application directe aux problèmes de micro-météorologie et de technologie et devraient clarifier de nombreux aspects des écoulements par convection naturelle qui apparaissent au-dessus de plans horizontaux et inclinés sous l'action d'une impulsion initiale de couches limites stratifiées instables prés de la surface.

GRENZSCHICHTSTRÖMUNG DURCH FREIE KONVEKTION AN WAAGERECHTEN UND LEICHT GENEIGTEN FLÄCHEN

Zusammenfassung—Es wird eine laminare Grenzschichtströmung betrachtet als Folge freier Konvektion an horizontalen und leicht geneigten Flächen. Die Wirkung einer geringen Flächenneigung auf Strömung und Wärmetransport wird untersucht als Störung der Strömung über einer waagerechten Fläche. Näherungslösungen höherer Ordnung wurden numerisch bestimmt für die Geschwindigkeits-, Druck- und Temperaturverteilungen mit einer Prandtl-Zahl 0,7. Diese Verteilungen werden auch für eine genau horizontale Fläche gegeben für einen weiten Bereich der Prandtl-Zahl. Sowohl die Bedingung gleichförmiger Temperatur als auch gleichförmigen Wärmestroms an der Heizfläche werden betrachtet. Zur Untersuchung der laminaren Grenzschicht wurde ein Mach–Zehnder Interferometer benutzt. Die Temperaturverteilungen und die Wärmeübertragungsparameter wurden zum ersten Mal in einer solchen Strömung für Luft bei Atmosphärendruck gemessen.

Die vorliegenden Ergebnisse kann man direkt auf mikrometeorologische und technologische Probleme anwenden. Sie sollten zur Klärung vieler Aspekte von freien Konvektionsströmungen beitragen, die an horizontalen und geneigten Ebenen auftauchen infolge von instabilen Fluidschichten nahe der Heizfläche.

ТЕЧЕНИЕ В ПОГРАНИЧНОМ СЛОЕ ПРИ ЕСТЕСТВЕННОЙ КОНВЕКЦИИ НАД ГОРИЗОНТАЛЬНЫМИ И СЛЕГКА НАКЛОНЕННЫМИ ПОВЕРХНОСТЯМИ

Аннотация—Рассматривается ламинарное течение в пограничном слое при естественной конвекции над горизонтальными и слегка наклоненными поверхностями. Влияние небольшого наклона поверхности на течение изучается как возмущение потока над горизонтальной поверхностью. Численно найдены аппроксимации высшего порядка для полей скорости, давления и температуры при $Pr = 0{,}7$. Эти поля приводятся также для идеально горизонтальной поверхности в широком диапазоне изменения числа Прандтля. Рассматриваются два типа условий на поверхности : постоянная температура и заданный

тепловой поток. Для изучения механизма ламинарного пограничного слоя использовался интерферометр Маха-Цендера. Впервые были измерены распределения температуры и параметры теплообмена в таком потоке и в воздухк при атмосферном давлении.

Данные результаты можно непосредственно использовать в микрометеорологических и технологических задачах. Они должны объяснить многие картины течений при естественной конвекции над горизонтальными и наклонными плоскостями, вызванные начальной неустойчивостью слоев жидкости вблизи поверхности.